Introduction to de Rham Cohomology Theory

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Final Report, May 2023

Outline

Table of Contents

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- Gives the name “differential” for (co)homology;
- Connects calculus, algebra and topology;
- De Rham cohomology is a powerful tool for studying smooth manifolds.

Calculus

[MT97, Chapter 1]. Fix an open set $U \subseteq \mathbb{R}^2$, and let $C^\infty(U, R^k)$ denote the set of smooth functions from U to $\mathbb{R}^k$.

Definition (Gradient, Rotation)

For $F \in C^\infty(U, \mathbb{R})$ and $(F_1, F_2) \in C^\infty(U, \mathbb{R}^2)$, we define

$$\operatorname{grad} F := (D_1 F, D_2 F), \operatorname{rot}(F_1, F_2) := D_1 F_2 - D_2 F_1.$$

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It follows that

Proposition

We have $\operatorname{rot} \circ \operatorname{grad} = 0$, *and the following complex.*

$$C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U, \mathbb{R}) \longrightarrow 0.$$

Definition

The first homology group is the quotient group is defined as follows: $H^1_{dR}(U) := \ker(\mathrm{rot})/\operatorname{Im}(\mathrm{grad})$.

Theorem

For a star-shaped open set $U \subseteq \mathbb{R}^2$, we have $H^1_{dR}(U) = 0$.

Sketch of Proof.

WLOG, let U be star-shaped with respect to 0. Let $\mathrm{rot}(f_1, f_2) = 0$. The function

$$F(x_1, x_2) := \int_0^1 x_1 f_1(tx_1, tx_2) + x_2 f_2(tx_1, tx_2) dt$$

satisfies $D_1 F = f_1, D_2 F = f_2$ whenever $D_2 f_1 = D_1 f_2$. Therefore, $\mathrm{grad}\, F = (f_1, f_2)$ and hence $\operatorname{Im}(\mathrm{grad}) = \ker(\mathrm{rot})$. □

Remark

It is not very simple to find other $H^1_{dR}(U)$, *even* $U = \mathbb{R}^2 \setminus \{0\}$.

Definition

For $k \geq 1$ and an open subset $U \subseteq \mathbb{R}^k$, we define

$$H^0_{dR}(U) := \ker(\text{grad}).$$

We will show that it counts the connected components of U (equivalently, path-connected components, since $\mathbb{R}^n$ is locally path-connected).

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Theorem

For an open subset $U \subseteq \mathbb{R}^k$, $H^0_{dR}(U) = \mathbb{R}$ *iff* U *is connected.*

Proof.

Sufficiency: we want $f \in H^0_{dR}(U)$ to be just c_f, where c_f is a constant for each f. Each $x_0 \in U$, there is an open neighbourhood $V(x_0)$ of x_0 such that $f = f(x_0)$ on $V(x_0)$. Then the set

$$\{x \in U \mid f(x) = f(x_0)\} = f^{-1}\big(f(x_0)\big)$$

is closed, since f is continuous. It is also open since $V(x_0) \subseteq U$. Now we prove what we wanted. Connectivity of U means that f is constant.
Necessity: if U is not connected, we can take $f\colon U \to \{0, 1\}$ that is smooth and surjective. Then $\dim H^0_{dR}(U) > 1$. □

There are relevant results, such as [Rot88, Exercise 12.2].

Exercise

If $\{X_\lambda \colon \lambda \in \Lambda\}$ is the set of path components of X, prove that, for every $n \geq 0$,

$$H^n(X; G) \cong \prod_\lambda H^n(X_\lambda; G).$$

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Similarly

Corollary

For open set $U \subseteq \mathbb{R}^k$, we have $H^0_{dR}(U) = \prod_{\pi_0(U)} \mathbb{R}$.

Table of Contents

Differential Forms

Let $x_1, \dots, x_n$ be the linear coordinates on $\mathbb{R}^n$. Then for any fixed point $p \in \mathbb{R}^n$, we can define n linear map

$$\begin{aligned} \partial/\partial x_i : C_p^\infty &\to \mathbb{R} \\ f &\mapsto D_i f \mid_p . \end{aligned}$$

The linear space spanned by $\{\partial/\partial x_i\}$ over $\mathbb{R}$ is denoted $T_p(\mathbb{R}^n)$, which is called the tangent space. The dual space $T_p^*(\mathbb{R}^n)$ is called the cotangent space. Let $\{dx_i\}$ to be the dual basis of $\{\partial/\partial x_i\}$.

For every fixed integer $1 \leq q \leq n$, let $\Omega^q(\mathbb{R}^n)$ to be the free-module of ring $C^\infty(\mathbb{R}^n)$ with the basis

$$\{dx_{i_1}...dx_{i_q} \mid 1 \leq i_1 < i_2 < ... < i_q \leq n\}.$$

Every element in $\Omega^q(\mathbb{R}^n)$ has the form

$$\sum_{1 \leq i_1 < i_2 < ... < i_q \leq n} f_{i_1 \ldots i_q} dx_{i_1} \ldots dx_{i_q},$$

where $f_{i_1 \ldots i_q} \in C^\infty(\mathbb{R}^n)$, we call it the C^∞ q-form.

Remark

We define $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Define $\Omega^* = \bigoplus \Omega^q(\mathbb{R}^n)$, then Ω^* is also a $C^\infty(\mathbb{R}^n)$-module. Define a multiple operation on Ω^* as usual sense but with relation:

$$\begin{cases} (dx_i)^2 = & 0 \\ dx_i dx_j = & -dx_j dx_i, i \neq j. \end{cases}$$

There is a differential operator

$$d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$$

defined as follows:

1. If $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum D_i f \ dx_i$;
2. If $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

Example

In the case $\mathbb{R}^2$: let $\omega_0 = f$ $\omega_1 = Pdx + Qdy$ and $\omega_2 = Fdxdy$. We have

$$\begin{aligned} d\omega_0 =& D_1 f dx + D_2 f dy, \\ d\omega_1 =& (D_1 P dx + D_2 P dy) \wedge dx + (D_1 Q dx + D_2 Q dy) \wedge dy \\ =& (D_1 Q - D_2 P) dxdy. \\ d\omega_2 =& 0. \end{aligned}$$

Proposition

We have $d \circ d = 0$ and hence a chain complex:

$$\Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n) \longrightarrow \cdots \longrightarrow \Omega^{n-1}(\mathbb{R}^n) \longrightarrow \Omega^n(\mathbb{R}^n) \longrightarrow 0$$

The de Rham Complex

Example

Here is the de Rham complex of $\mathbb{R}^2$, isomorphic to the lower complex mentioned before:

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^2) & \xrightarrow{\quad d \quad} & \Omega^1(\mathbb{R}^2) & \xrightarrow{\quad d \quad} & \Omega^2(\mathbb{R}^2) & \xrightarrow{\quad d \quad} & 0 \\
\big\|\wr & & \big\|\wr & & \big\|\wr & & \\
C^\infty(\mathbb{R}^2, \mathbb{R}) & \xrightarrow{\text{grad}} & C^\infty(\mathbb{R}^2, \mathbb{R}^2) & \xrightarrow{\text{rot}} & C^\infty(\mathbb{R}^2, \mathbb{R}) & \longrightarrow & 0
\end{array}.$$

Example (continued)

Here isomorphisms are

$$\begin{array}{ccc} f & F_1 dx + F_2 dy & F dxdy \\ \downarrow & \downarrow & \downarrow \\ f & (F_1, F_2) & F \end{array}.$$

And d's are

$$f \longmapsto D_1 f dx + D_2 f dy, F_1 dx + F_2 dy \longmapsto (D_1 F_2 - D_2 F_1) dxdy.$$

De Rham Cohomology Groups

As we define homology groups of a complex:

Definition

The *de Rham cohomology groups* $H^k_{dR}(M)$ of a smooth manifold M are the quotient groups

$$H^k_{dR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Remark

The element of $\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$ *is called the closed k-form.*
The element of $\operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$ *is called the exact k-form.*

Table of Contents

Examples

Example

For any smooth manifold M,

$$H^0_{dR}(M) \cong \mathbb{R}^k$$

where $k := \sharp\pi_0(M)$ is the number of connected (equivalently, path-connected) components of M.

It generalizes the corollary mentioned before:

Corollary

For open set $U \subseteq \mathbb{R}^k$, we have $H^0_{dR}(U) = \prod_{\pi_0(U)} \mathbb{R}$.

The Poincaré Lemma

Theorem (Poincaré's Lemma)

Let $U \subseteq \mathbb{R}^n$ be a star-shaped open set. Then

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

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Example

Let $n \geq 1$ be fixed. For $\mathbb{R}^n$, we have

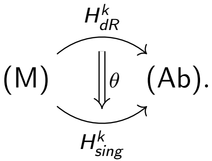
$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

The case $k = 0$ follows from the previous example and the case $(k, n) = (1, 2)$ was proved in Section 1. General proof can be founded in [MT97, Theorem 3.15].

De Rham's Theorem

Theorem (De Rham)

Let (M) *be the category of smooth, connected, orientable manifolds with smooth maps. There is a natural isomorphism:*



- Connects de Rham cohomology to classical cohomology.
- Gives an intepretation of why de Rham cohomology is a topological invariant.

Other Important Results

- *Functoriality* of Ω^*: in fact, we have a contravariant functor Ω^* from the category (SmoothEuclid) to (ComDiffAlg). [BT13, Chapter 1.2]
- *Mayer-Vietoris sequence* for de Rham cohomology, which contributes one way to calculate homology groups of the punctured plane $\mathbb{R}^2 \setminus \{0\}$, spheres and so on. [MT97, Chapter 6].
- De Rham cohomology with *compact supports*. [BT13, Chapter 1.1]
- De Rham cohomology *ring*: $H^\bullet(M) := \bigoplus_{k\in\mathbb{Z}} H^k(M)$ is naturally a commutative graded ring (w.r.t the wedge product as a special case of cup product). e.g., $H^\bullet(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}[X]/(X^2)$. [Mun18, Chapter 5]

Thank you!

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